

SAut(F_n) CANNOT ACT ON SMALL TORI

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ABSTRACT. We study smooth actions of $\text{SAut}(F_n)$, the unique subgroup of index two in the automorphism group of a free group of rank n , as a part of the generalized 'Zimmer program'. In particular, we show that every action of $\text{SAut}(F_n)$ on a low dimensional torus is trivial.

1. INTRODUCTION

In the mathematical world, this work is located in the area of geometric group theory, a field at the intersection of algebra, geometry and topology. Geometric group theory studies the interaction between algebraic and geometric properties of groups. One is interested to understand on which 'nice' geometric spaces a given group can act in a reasonable way and how geometric properties of these spaces are reflected in the algebraic structure of the group. Here, the spaces will be tori, while the group will be $\text{SAut}(F_n)$.

In this work we study smooth actions of $\text{SAut}(F_n)$ as a part of the generalized 'Zimmer program' whose aim is to understand actions of large groups on compact manifolds, see e.g. [FF11]. Given a group whose low dimensional linear representations are trivial, one wants to know whether all group representations into $\text{Diff}(M)$, where M is a low dimensional compact manifold, are also trivial.

To be precise, let \mathbb{Z}^n be the free abelian group and F_n the free group of rank n . One goal for a group theorist is to understand the structure of their automorphism groups, $\text{GL}_n(\mathbb{Z})$ resp. $\text{Aut}(F_n)$. The abelianization map $F_n \rightarrow \mathbb{Z}^n$ gives a natural epimorphism $\text{Aut}(F_n) \twoheadrightarrow \text{GL}_n(\mathbb{Z})$. The special automorphism group of F_n , which we will denote by $\text{SAut}(F_n)$, is defined as the preimage of $\text{SL}_n(\mathbb{Z})$ under this map. Much of the work on $\text{Aut}(F_n)$ and $\text{SAut}(F_n)$ is motivated by the idea that $\text{GL}_n(\mathbb{Z})$ and $\text{Aut}(F_n)$ resp. $\text{SL}_n(\mathbb{Z})$ and $\text{SAut}(F_n)$ should have many properties in common. Here we follow this idea and present analogies between these groups with respect to smooth actions on low dimensional tori.

It was shown by WEINBERGER in [We93] that any smooth action of $\text{SL}_n(\mathbb{Z})$ on the d -dimensional torus $\mathbb{T}^d = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ is trivial if $d < n$. This bound on the dimension of the torus is sharp, as $\text{SL}_n(\mathbb{Z})$ admits a linear faithful action on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$. We prove an analogue of WEINBERGER's result for smooth actions of $\text{SAut}(F_n)$.

Theorem. *Let $n \geq 3$ and $\Phi : \text{SAut}(F_n) \rightarrow \text{Diff}(\mathbb{T}^d)$ be a smooth action. If $d < n$, then Φ is trivial.*

Combining one result of WEINBERGER concerning smooth actions of non-abelian finite groups on tori with results of BRIDSON and VOGTMANN about strong constraints on homomorphisms from $\text{SAut}(F_n)$, we prove the above theorem.

2. THE AUTOMORPHISM GROUP OF A FREE GROUP

As the main protagonist in this work is the group $\text{SAut}(F_n)$. We start with the definition of this group, and establish some notation to be used throughout.

We begin with the definition of the automorphism group of the free group of rank n . Let F_n be the free group of rank n with a fixed basis $X := \{x_1, \dots, x_n\}$. We denote by $\text{Aut}(F_n)$ the automorphism group of F_n and by $\text{SAut}(F_n)$ the unique subgroup of index two in $\text{Aut}(F_n)$.

Let us first introduce a notations for some elements of $\text{Aut}(F_n)$. We define involutions (x_i, x_j) for $i, j = 1, \dots, n$, $i \neq j$ as follows:

$$(x_i, x_j)(x_k) := \begin{cases} x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \\ x_k & \text{if } k \neq i, j. \end{cases}$$

In particular, the alternating group Alt_n is a subgroup of $\text{SAut}(F_n)$.

The following variant of a result by BRIDSON and VOGTMANN [BV11, 3.1] will be used here to prove that certain actions on spaces of $\text{SAut}(F_n)$ are indeed trivial. For a detailed proof the reader is referred to [Va10, 1.13].

Proposition 2.1. *Let $n \geq 3$, G be a group and $\phi : \text{SAut}(F_n) \rightarrow G$ a group homomorphism. If there exists $\alpha \in \text{Alt}_n - \{\text{id}_{F_n}\}$ with $\phi(\alpha) = 1$, then ϕ is trivial.*

It was proven by BRIDSON and VOGTMANN in [BV11, 1.1] that if $n \geq 3$ and $d < n$, then $\text{SAut}(F_n)$ cannot act non-trivially by homeomorphisms on any contractible manifold of dimension d . Hence

Proposition 2.2. *Let $n \geq 3$ and $\rho : \text{SAut}(F_n) \rightarrow \text{SL}_d(\mathbb{R})$ be a linear representation. If $d < n$, then ρ is trivial.*

In [Va15] we proved in a purely group theoretical way the above proposition.

We note that the group $\text{SAut}(F_n)$ is perfect, therefore the image of a linear representation of $\text{SAut}(F_n)$ is a subgroup of $\text{SL}_d(\mathbb{R})$.

3. PROOF OF MAIN THEOREM

We begin the proof of main theorem by the following proposition by WEINBERGER. The proof of this result is decidedly smooth. First, we note that the first singular cohomology group of a torus $H^1(\mathbb{T}^d; \mathbb{Z})$ is isomorphic to \mathbb{Z}^d .

Proposition 3.1. ([We93, 2]) *Let G be a non-abelian finite group and $\psi : G \rightarrow \text{Diff}(\mathbb{T}^d)$ a smooth action. If*

$$\begin{aligned} H^1(\psi) : G &\rightarrow \text{GL}_d(\mathbb{Z}), \\ g &\mapsto H^1(\psi(g)) : H^1(\mathbb{T}^d; \mathbb{Z}) \rightarrow H^1(\mathbb{T}^d; \mathbb{Z}) \end{aligned}$$

is trivial, then ψ is not injective.

Now we have all the ingredients to prove

Theorem 3.2. *Let $n \geq 3$ and $\Phi : \text{SAut}(F_n) \rightarrow \text{Diff}(\mathbb{T}^d)$ be a smooth action. If $d < n$, then Φ is trivial.*

Proof. According to Proposition 2.2, the following action is trivial for $d < n$:

$$\begin{aligned} \iota \circ H^1(\Phi) : \text{SAut}(F_n) &\rightarrow \text{SL}_d(\mathbb{Z}) \hookrightarrow \text{SL}_d(\mathbb{R}) \\ \alpha &\mapsto H^1(\Phi(\alpha)) \mapsto H^1(\Phi(\alpha)). \end{aligned}$$

In particular, the map $H^1(\Phi)$ is trivial.

We consider the subgroup Alt_n in $\text{SAut}(F_n)$ and the restriction of Φ to this group. The map $H^1(\Phi|_{\text{Alt}_n})$ is trivial, therefore by Proposition 3.1 the map $\Phi|_{\text{Alt}_n}$ is not injective and by Proposition 2.1 it follows that Φ is trivial. \square

Remark 3.3. *Note that this bound on the dimension of the torus is sharp, as $\text{SL}_n(\mathbb{Z})$ admits a linear faithful action on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$. Therefore $\text{SAut}(F_n)$ admits a smooth non-trivial action on the flat torus as well:*

$$\text{SAut}(F_n) \twoheadrightarrow \text{SL}_n(\mathbb{Z}) \rightarrow \text{Diff}(\mathbb{R}^n/\mathbb{Z}^n).$$

REFERENCES

- [BV11] M. Bridson and K. Vogtmann, Actions of automorphism groups of free groups on homology spheres and acyclic manifolds, *Comment. Math. Helv.* 86 (1), (2011).
- [FF11] B. Farb and D. Fisher, *Geometry, Rigidity, and Group Actions*, University of Chicago Press, Chicago, IL (2011).
- [Va10] O. Varghese, Wirkungen von Aut(F_n), Diploma Thesis at WWU (2010).
- [Va15] O. Varghese, Low dimensional linear representations of SAut(F_n), arXiv:1509.00187.
- [We93] S. Weinberger, $SL_n(\mathbb{Z})$ cannot act on small tori, *Geometric Topology* (1993).

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